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Rank one perturbations of Jacobi matrices with mixed spectra

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Let A be a self-adjoint operator and φ its cyclic vector. In this work we study spectral properties of rank one perturbations of A

$$A_\theta = A + \theta\langle\varphi, \cdot\rangle\varphi$$

in relation to their dependence on the real parameter θ. We find bounds on averages of spectral measures for semi-infinite Jacobi matrices and give criteria which guarantee existence of mixed spectral types for θ in a set of positive Lebesgue measure.

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1 Introduction

In this paper, spectral properties of rank one perturbations of a fixed semi-infinite Jacobi operator are considered. These perturbations depend on a real parameter θ (coupling constant) and our main concern is the behavior of different spectral types as this parameter θ varies. In particular, we address the problem of coexistence of distinct kinds of spectra for sets of θ's of positive Lebesgue measure. It is known that if the support of one type of spectrum for the unperturbed operator has positive measure, then there will be a set of θ's of positive measure too, such that the corresponding perturbed operator A will have this kind of spectrum somewhere in the real line $\mathbb{R}$. Thus, a support of the spectrum with positive measure gives rise to a set of coupling constants of positive measure. This result straightforwardly follows from the averaging formula (2.1) below.

Our aim is to find out more about the structure of the set of coupling constants, particularly about its distribution on the real line. For the case of semi-infinite Jacobi matrices with potentials vanishing in a finite interval, we were able to give conditions on the size of intervals of coupling constants which imply existence of a set of positive measure contained in such intervals, so that the operators generated by rank one perturbations corresponding to points in that set will have mixed singular and a.c. spectra. These conditions will depend on the measure of the support of the absolutely continuous part and on the length of the interval where the potential of the unperturbed operator vanishes. The main tools we use are bounds on averages of spectral measures, which are associated with self-adjoint operators generated by rank one perturbations. For the bound from below, we need a result of Chebyshev et al. which follows from the quadrature formula.

The paper is organized as follows. In Section 2 we prove some preliminary results which give us the required bounds and use them for general Jacobi operators. It is shown that the average of norms of eigenvectors for a finite part of the matrix associated to the operator is bounded by the Lebesgue measure of subsets contained in the support of the a.c. part of the spectral measure. As a corollary, bounds on the average of these norms are given in terms of the difference of eigenvalues. In Section 3 we consider operators which are constant in an interval. It will be important to construct finite matrices with eigenvalues that do not depend on a real parameter. Explicit

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estimates are given on the measure of sets where the spectral density is positive and conditions are obtained which imply mixed spectra. Examples are discussed and it is shown that counterexamples are possible, so that the mentioned coexistence does not hold in every case. These results are related to the ones in [7] where the Sturm–Liouville case was studied. In the last section we turn again to the general case and give simple criteria which imply existence of singular spectrum for rank one perturbations.

2 Preliminary results and bounds for general Jacobi operators

We consider rank one perturbations of the self-adjoint operator A in the Hilbert space H

$$A_\theta \;=\; A + \theta\langle\varphi,\cdot\rangle\varphi\,,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in H, φ is a cyclic vector of A, and $\theta \in \mathbb{R}$. We shall denote by ρ_θ the measure generated by the spectral measure of A_θ and the cyclic vector φ, so that

$$F_\theta(z) \;:=\; \left\langle\varphi,(A_\theta - z)^{-1}\varphi\right\rangle \;=\; \int_{\mathbb{R}} \frac{\mathrm{d}\rho_\theta(x)}{x-z}\,.$$

We set $F = F_0$.

It turns out that for any Borel set B

$$\int_{\mathbb{R}} \rho_\theta(B)\,\mathrm{d}\theta \;=\; |B| \tag{2.1}$$

(see [13, Theorem 1.8] or [11]), where $|\cdot|$ denotes Lebesgue measure. When the integration takes place over the interval (α,β) then the following result holds (see [13, Theorem 1.12]).

Lemma 2.1 *Let $\alpha < \beta$. Then if B is a Borel set, we have*

$$\int_\alpha^\beta \rho_\theta(B)\,\mathrm{d}\theta \;=\; \frac{1}{\pi}\int_B \arg\left(\frac{1+\beta F(E+i0)}{1+\alpha F(E+i0)}\right)\mathrm{d}E\,.$$

Now, let us define the set

$$\Lambda_M \;=\; \{E : \operatorname{Im} F(E+i0) > M\}\,, \quad M \;\geq\; 0\,, \tag{2.2}$$

where

$$F(E+i0) \;:=\; \lim_{y\downarrow 0} F(E+iy)$$

which exists and is finite for a.e. E.

Lemma 2.1 yields the following result:

Lemma 2.2 *Let $\alpha < \beta$ such that $\alpha\beta > 0$. Then for any interval I the following inequality holds*

$$\int_\alpha^\beta \rho_\theta(I\cap\Lambda_M)\,\mathrm{d}\theta \;\leq\; \frac{2}{\pi}\,\arctan\!\left(\frac{1}{2M}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right)|I\cap\Lambda_M|\,.$$

P r o o f. Let

$$w \;=\; Tz \;=\; \frac{\beta z + 1}{\alpha z + 1}\,.$$

For each $M > 0$, T maps the half plane $\operatorname{Im} z > M$ onto the disk

$$\left(\frac{\beta}{\alpha} - x\right)^2 + \left(y - \frac{\beta-\alpha}{2\alpha^2 M}\right)^2 \;<\; \left(\frac{\beta-\alpha}{2\alpha^2 M}\right)^2\,.$$

From here it follows that if $\operatorname{Im} z > M$ then $\arg w \leq 2\arctan\left(\frac{1}{2M}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right)$.

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Using Lemma 2.1 we obtain

$$\begin{aligned}\int_\alpha^\beta \rho_\theta(I\cap\Lambda_M)\,\mathrm{d}\theta &= \frac{1}{\pi}\int_{I\cap\Lambda_M} \arg T(F(E+i0))\,\mathrm{d}E\\ &\le \frac{2}{\pi}\int_{I\cap\Lambda_M} \arctan\left(\frac{1}{2M}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right)\mathrm{d}E\\ &= \frac{2}{\pi}\arctan\left(\frac{1}{2M}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right)|I\cap\Lambda_M|\,. \end{aligned}$$

□

The estimate given in Lemma 2.2, holds in general for self-adjoint operators generated by rank one perturbations.

Remark 2.3 To study the singular spectrum we only need to consider $M=0$, in which case Lemma 2.2 is not required since we can use (2.1). Nevertheless, we believe that the more general situation $M\ge 0$ could be of interest, particularly since the set Λ_M is related to asymptotic properties of the generalized eigenvectors (cf. [12]). Moreover, the proofs do not become more complicated if we take $M\ge 0$.

Throughout this paper the unperturbed self-adjoint operator will be considered to be a Jacobi operator. In the Hilbert space of square summable sequences $\ell^2\big(\{0,1,2,\dots\}\big)$, we define a Jacobi operator J as the closed symmetric operator whose matrix representation is a Jacobi matrix, i. e.,

$$\begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots\\ b_0 & a_1 & b_1 & 0 & \cdots\\ 0 & b_1 & a_2 & b_2 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{2.3}$$

where $\{a_n\}$ is a sequence of real numbers and $\{b_n\}_{n=0}^\infty$ is a sequence of real positive numbers. We assume that the limit point case holds at ∞ and therefore that the operator J is the only self-adjoint realization of the matrix (2.3), see [14] and [3, Chapter VII] for a review of these concepts. By the spectral theorem there is a spectral measure ρ for J corresponding to the vector $\delta_0=(1,0,0,\dots)$ such that for any bounded function f of J

$$\langle\delta_0, f(J)\delta_0\rangle = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\rho(\lambda)\,.$$

Now, consider the following finite Jacobi matrix

$$J_N := \begin{pmatrix} a_0 & b_0 & & & & \bigcirc\\ b_0 & a_1 & b_1 & & & \\ & b_1 & a_2 & b_2 & & \\ & & & \ddots & & \\ \bigcirc & & & & a_{N-2} & b_{N-2}\\ & & & & b_{N-2} & a_{N-1} \end{pmatrix} \tag{2.4}$$

and let us denote by $\vec{P}(\lambda_j)=\big(P_0(\lambda_j),P_1(\lambda_j),\dots,P_{N-1}(\lambda_j)\big)^T$ the eigenvectors of J_N corresponding to the eigenvalues λ_j, $j=1,\dots,N$. We shall assume the vector $\vec{P}(\lambda_j)$ to be normalized in such a way that $P_0(\lambda_j)=1$ for all j, and that $\lambda_1<\ldots<\lambda_N$.

A notable consequence of the quadrature formula is the following result, firstly formulated by Chebyshev in a more general form and proved independently by Markov and Stieltjes (see [1, Theorem 2.5.4] and [8]). We include a sketch of the proof for the readers' convenience.

Theorem 2.4 *Let ρ be the spectral measure with respect to δ_0 of a self-adjoint Jacobi operator of the form* (2.3) *with fixed given entries $a_o,\dots,a_{N-2},b_0,\dots,b_{N-2}$. Let*

$$\big\|\vec{P}(\lambda_\ell)\big\| = \left(\sum_{n=0}^{N-1} P_n^2(\lambda_\ell)\right)^{1/2}$$

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where $\vec{P}(\lambda_\ell)$ are the eigenvectors of the matrix J_N corresponding to the eigenvalues λ_ℓ as defined above. Then the following inequality holds:

$$\rho\big((\lambda_j, \lambda_{i+1})\big) \;\ge\; \sum_{\ell=j+1}^{i} \big\|\vec{P}(\lambda_\ell)\big\|^{-2}, \quad j \in \{1, \dots, N-2\} \quad \textit{and} \quad j \,<\, i \,\le\, N-1\,.$$

P r o o f. (Sketch). A particular case of the quadrature formula (see [1]) valid for every polynomial $\varphi(t)$ of degree no greater than $2N-2$ is

$$\int_{\mathbb{R}} \varphi(t)\,\mathrm{d}\rho(t) \;=\; \sum_{k=1}^{N} \frac{\varphi(\lambda_k)}{\big\|\vec{P}(\lambda_k)\big\|^2}\,, \tag{2.5}$$

where $\vec{P}(\lambda_k)$ is the eigenvector of the finite matrix J_N corresponding to the eigenvalue λ_k and ρ is the spectral measure of the semi-infinite Jacobi matrix J.

Consider now the polynomial $\varphi(t)$ to be defined as follows

$$\varphi(\lambda_k) \;=\; \begin{cases} 1 & \text{if} \quad k \,=\, j+1\,, \dots, i\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.6}$$

$$\varphi'(\lambda_k) \;=\; 0 \quad \text{for all} \quad k \,\neq\, i+1\,. \tag{2.7}$$

From (2.5) and (2.6) it follows straightforwardly that

$$\int_{\mathbb{R}} \varphi(t)\,\mathrm{d}\rho(t) \;=\; \sum_{k=j+1}^{i} \big\|\vec{P}(\lambda_k)\big\|^{-2}\,.$$

Now, it is not difficult to show that conditions (2.6) and (2.7) imply

$$\varphi(t) \in \begin{cases} (0,1] & \text{if} \quad \lambda_j \,\le\, t \,<\, \lambda_{i+1}\,, \\ (-\infty, 0] & \text{if} \quad t \,\ge\, \lambda_{i+1}\,, \end{cases}$$

and from this one obtains

$$\int_{\mathbb{R}} \varphi(t)\,\mathrm{d}\rho(t) \;\le\; \int_{\lambda_j+0}^{\lambda_{i+1}-0} \mathrm{d}\rho(t)\,.$$

□

Remark 2.5 Notice that the assertion of the theorem holds even when the semi-infinite matrix J does not have the same entry a_{N-1} as J_N.

Remark 2.6 The quadrature formula (2.5) is actually valid for every measure that is the solution of the moment problem associated with the semi-infinite Jacobi matrix J. Hence, Theorem 2.4 holds independently of whether the matrix J corresponds to the limit point case.

Using the estimate from above given by Lemma 2.2 and the estimate from below given by Theorem 2.4 we can get bounds on the averages of norms of eigenvectors of J_N that hold for general Jacobi operators J.

In the theorem below, ρ_{a_0} denotes the spectral measure of the Jacobi operator J with matrix representation (2.3) which has first entry a_0. Similarly, $\lambda_j(a_0)$ denotes an eigenvalue of the finite matrix J_N defined above with first entry a_0 and $\vec{P}\big(\lambda_j(a_0)\big)$ the corresponding eigenvector. As in the previous theorem the norm of these vectors will be denoted by $\big\|\vec{P}\big(\lambda_j(a_0)\big)\big\|$.

Theorem 2.7 *Let $\alpha < \beta$. Denote by λ_j the eigenvalues of the finite matrix J_N with $a_0 = 0$ and define $I = \big(\lambda_{j-2}, \lambda_{j+2}\big)$.*

If

$$\int_{\alpha}^{\beta} \rho_{a_0}\big(I \cap \Lambda_M^C\big)\,\mathrm{d}a_0 \;=\; 0\,,$$

then

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$$\int_\alpha^\beta \left\|\vec{P}\big(\lambda_j(a_0)\big)\right\|^{-2} \mathrm{d}a_0 \;\le\; \frac{2}{\pi}\,|I\cap\Lambda_M| \int_0^{\frac{1}{2M}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)} \frac{\mathrm{d}y}{y^2+1} \quad \textit{for} \quad 3 \le j \le N-2\,.$$

Recall that $|\cdot|$ denotes Lebesgue measure and Λ_M is defined by (2.2).

Proof. Let us denote by $J_N(a_0)$ the finite matrix (2.4) with first entry a_0. Observe that it is not possible for a real number λ to be simultaneously an eigenvalue for $J_N(a_0)$ and $J_N(\tilde{a}_0)$ with $a_0 \neq \tilde{a}_0$. To see this consider the recurrence formulas generated by J_N

$$\begin{aligned}(a_{N-1}-\lambda)w_{N-1} + w_{N-2}b_{N-1} &= 0\,,\\ b_i w_{i-1} + (a_i-\lambda)w_i + b_{i+1}w_{i+1} &= 0\,, \quad 1 \le i \le N-2\,.\end{aligned}$$

If we fix $w_{N-1} \in \mathbb{R}$, then $w_{N-2}, w_{N-3}, \dots, w_0$ are defined by these equations and $w_{N-k}(\lambda)$ will be a polynomial of degree $k-1$ in λ.

Define

$$w_{-1}(\lambda) \;=\; (a_0-\lambda)w_0 + b_1 w_1\,,$$

where $w_{-1}(\lambda)$ is a polynomial of degree N in λ and the roots of this polynomial are the eigenvalues of J_N. Since the coefficient a_0 does not appear in w_0 or w_1, it is not possible to have the same eigenvalue λ of J_N for two different values of a_0.

Therefore, for every a_0,

$$\big(\lambda_{j-1}(a_0), \lambda_{j+1}(a_0)\big) \subset \big(\lambda_{j-2}, \lambda_{j+2}v\big) \;=\; I$$

and

$$\rho_{a_0}(I) \;\ge\; \rho_{a_0}\big(\lambda_{j-1}(a_0), \lambda_{j+1}(a_0)\big)\,.$$

Hence, if we apply Theorem 2.4 and integrate we obtain

$$\int_\alpha^\beta \rho_{a_0}(I)\,\mathrm{d}a_0 \;\ge\; \int_\alpha^\beta \left\|\vec{P}\big(\lambda_j(a_0)\big)\right\|^{-2} \mathrm{d}a_0\,. \tag{2.8}$$

Now, assuming

$$\int_\alpha^\beta \left\|\vec{P}\big(\lambda_j(a_0)\big)\right\|^{-2} \mathrm{d}a_0 \;>\; \frac{2}{\pi}\,|I\cap\Lambda_M| \int_0^{\frac{1}{2M}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)} \frac{\mathrm{d}y}{y^2+1} \tag{2.9}$$

and putting together inequalities (2.8), (2.9) and Lemma 2.2 we obtain

$$\int_\alpha^\beta \rho_{a_0}(I)\,\mathrm{d}a_0 \;>\; \int_\alpha^\beta \rho_{a_0}\big(I\cap\Lambda_M\big)\,\mathrm{d}a_0\,,$$

which is a contradiction to the hypotheses

$$\int_\alpha^\beta \rho_{a_0}\big(I\cap\Lambda_M^C\big)\,\mathrm{d}a_0 \;=\; 0\,.$$

Observe that the above theorem holds if we consider a_{N-1} instead of a_0. □

Remark 2.8 If we are able to have estimates on the averages of the norms of the eigenvectors $P\big(\lambda_j\big)$ of J_N and on the Lebesgue measure of the set Λ_M then we can use Theorem 2.7 to obtain conditions which imply

$$\int_\alpha^\beta \rho_{a_0}\big(I\cap\Lambda_M^C\big)\,\mathrm{d}a_0 \;>\; 0\,,$$

that is, $\rho_{a_0}\big(I\cap\Lambda_M^C\big) > 0$ for $a_0 \in B \subset (\alpha,\beta)$ and $|B| > 0$. This has particular interest in the case $M = 0$ since Λ_0^C is a minimal support of the singular spectra. We shall have mixed spectrum in I for a positive measure set of the a_0's in (α,β) if $\int_\alpha^\beta \rho_{a_0}\big(I\cap\Lambda_0^C\big)\,\mathrm{d}a_0 > 0$ and $|I\cap\Lambda_0| > 0$, Λ_0 dense in I.

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As a consequence of the previous theorem we obtain the following estimate.

Corollary 2.9 *Consider the finite matrix J_N defined in* (2.4) *and let λ_j, $j = 1, \dots, N$, and $\vec{P}(\lambda_j)$ denote its eigenvalues and the corresponding eigenvectors. Then*

$$\int_{\mathbb{R}} \big\|\vec{P}_j(a_0)\big\|^{-2}\,\mathrm{d}a_0 \;\le\; \lambda_{j+2} - \lambda_{j-2}\,, \quad \textit{where} \quad 3 \,\le\, j \,\le\, N-2\,.$$

P r o o f. Take a semi-infinite Jacobi matrix such that $|I \cap \Lambda_0| = |I|$ where $I = \lambda_{j+2} - \lambda_{j-2}$. We can, for example, start with a spectral function ρ such that for almost every $x \in I$, $\frac{\mathrm{d}\rho}{\mathrm{d}x}(x) > 0$ and then construct the operator J as in Example 3.4 (b) following Theorem 3.2.

If we change a_i, $i = 0, \dots, N-1$, and b_i, $i = 0, \dots, N-2$, then we still have for the perturbed operator that $|I \cap \Lambda_0| = |I|$ holds. This follows from the Kato–Rosenblum theorem about the invariance of the absolutely continuous part under trace class perturbations [10] or the Gilbert–Pearson theory of subordinacy [9], since a local perturbation does not modify the set of points where we do not have subordinate solutions.

Since $|I \cap \Lambda_0| = |I|$ implies $\big|I \cap \Lambda_0^C\big| = 0$ and $\int_\alpha^\beta \rho_{a_0}\big(I \cap \Lambda_0^C\big)\,\mathrm{d}a_0 \le \big|I \cap \Lambda_0^C\big| = 0$, we can apply the previous theorem to obtain

$$\int_\alpha^\beta \big\|\vec{P}\big(\lambda_j(a_0)\big)\big\|^{-2}\,\mathrm{d}a_0 \;\le\; |I| \;=\; \lambda_{j+2} - \lambda_{j-2}\,. \qquad \square$$

3 Jacobi operators constant in an interval and examples

In what follows we shall consider the case where the potential of J is constant in an interval, that is $a_0 = a_1 = \dots = a_{N-1} =: a$, and where $b_0 = b_1 = \dots = b_{N-1} = 1$, since in this case quite explicit calculations are possible. A similar construction can be made for any family of finite matrices which preserve at least three eigenvalues when the first entry a_0 and the last one, a_{N-1}, vary, provided an estimation of the norm of eigenvectors is possible.

Let us consider the linear transformation $J_N^\theta : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$J_N^\theta \;:=\; \begin{pmatrix} \theta & 1 & & & & \\ 1 & a & 1 & & \bigcirc & \\ & 1 & a & 1 & & \\ & & 1 & \ddots & & \\ \bigcirc & & & & \ddots & 1 \\ & & & & 1 & \frac{1}{\theta - a} + a \end{pmatrix}, \quad \theta\,, a \in \mathbb{R}\,. \tag{3.1}$$

The eigenvectors and the corresponding eigenvalues are

$$\begin{pmatrix} 1 \\ \theta^{-1} \\ \theta^{-2} \\ \vdots \\ \theta^{-(N-1)} \end{pmatrix} \longleftrightarrow \lambda_N \;=\; \theta + \frac{1}{\theta} + a\,,$$

$$\vec{P}\big(\lambda_j\big) \;:=\; \begin{pmatrix} P_0\big(\lambda_j\big) \\ P_1\big(\lambda_j\big) \\ \vdots \\ P_{N-1}\big(\lambda_j\big) \end{pmatrix} \longleftrightarrow \lambda_j \;=\; 2\cos(\pi(1 - j/N)) + a\,,$$

where

$$P_n\big(\lambda_j\big) \;:=\; \frac{\sin(\pi(1-j/N)(n+1)) - \theta\sin(\pi(1-j/N)n)}{\sin(\pi(1-j/N))}\,, \quad j \,=\, 1\,,2\,,\dots\,,N-1\,, \quad n \,=\, 0\,,1\,,2\,,\dots\,,N-1\,.$$

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We have normalized so that $P_0(\lambda_j) = 1$. One way of finding the eigenvectors above, is to look for solutions of the form

$$P_n(\lambda) \;=\; Az^n + Bz^{-n}$$

for complex z where the eigenvalues λ will be related to z by $\lambda = \left(z + \frac{1}{z}\right)$.

Notice that all eigenvalues of J_N^θ with the exception of one, do not depend on θ. This will be relevant for what follows. For simplicity we shall consider the case $a = 0$.

Let us calculate the norm of the eigenvector $\vec{P}(\lambda_j)$ corresponding to the eigenvalue λ_j which does not depend on θ. Using the identity $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, after some elementary calculations we obtain, for all $j = 1, \ldots, N-1$,

$$\|\vec{P}(\lambda_j)\|^2 \;=\; \sum_{n=0}^{N-1} P_n^2(\lambda_j) \;=\; \frac{N}{2(\sin\pi(1-j/N))^2}\left|e^{i\pi(1-j/N)} - \theta\right|^2 . \tag{3.2}$$

First we shall obtain a lower bound given by the next lemma.

Lemma 3.1 *Let J be a semi-infinite Jacobi matrix of the form* (2.3) *such that $a_i = 0$ and $b_i = 1$ for $i = 0, 1, 2, \ldots, N-2$, i.e.,*

$$J \;=\; \begin{pmatrix} 0 & 1 & & & & & & & \\ 1 & 0 & 1 & & & & & & \bigcirc \\ & \cdots & & & & & & & \\ & & \cdots & & & & & & \\ & & & \cdots & & & & & \\ & & & 1 & 0 & 1 & & & \\ & & & & 1 & a_{N-1} & b_{N-1} & & \\ & & & & & b_{N-1} & a_N & b_N & \\ & & & & & & \cdots & & \\ \bigcirc & & & & & & & \cdots & \\ & & & & & & & & \cdots \end{pmatrix} .$$

Consider the interval

$$I \;=\; \bigl(2\cos\bigl(\pi\bigl(1-(j-1)/N\bigr)\bigr),\; 2\cos\bigl(\pi\bigl(1-(j+1)/N\bigr)\bigr)\bigr),$$

assuming $j \in \{1, \ldots, N-1\}$ fixed. Let ρ_θ be the spectral measure of the operator

$$J_\theta \;=\; J + \theta\langle\delta_0, \cdot\rangle\delta_0 ,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in ℓ^2 and δ_0 is the sequence whose first element is 1 and all others are 0. Then

$$\int_\alpha^\beta \rho_\theta(I)\,\mathrm{d}\theta \;\geq\; \frac{2}{N}\sin(\pi(1-j/N))\int_\xi^\eta \frac{\mathrm{d}y}{y^2+1} ,$$

where $\xi = \frac{\alpha - \cos(\pi(1-j/N))}{\sin(\pi(1-j/N))}$ and $\eta = \frac{\beta - \cos(\pi(1-j/N))}{\sin(\pi(1-j/N))}$.

P r o o f. Consider the perturbed operator J_θ and apply Theorem 2.4. Then we know that

$$\rho_\theta\bigl((\lambda_{j-1}, \lambda_{j+1})\bigr) \;\geq\; \|\vec{P}(\lambda_j)\|^{-2} .$$

In our case, since $a_0, \ldots, a_{N-2}$ and $b_0, \ldots, b_{N-2}$ coincide with the entries of J_N^θ see (3.1), we can take λ_j and $\vec{P}_j(\lambda_j)$ to be the eigenvalues and the corresponding eigenvectors of J_N^θ. Clearly $I = \bigl(\lambda_{j-1}, \lambda_{j+1}\bigr)$.

According to (3.2) we have

$$\begin{aligned}\int_\alpha^\beta \rho_\theta\bigl((\lambda_{j-1}, \lambda_{j+1})\bigr)\,\mathrm{d}\theta \;&\geq\; \frac{2\sin^2(\pi(1-j/N))}{N}\int_\alpha^\beta \frac{\mathrm{d}\theta}{\left|e^{i\pi(1-j/N)} - \theta\right|^2} \\ &=\; \frac{2}{N}\,\sin\bigl(\pi(1-j/N)\bigr)\int_\xi^\eta \frac{\mathrm{d}y}{y^2+1} .\end{aligned}$$

□

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Notice that the operator J_θ is the same as J, just with a change in the first entry of the main diagonal.

With the tools developed in last section it is not hard to prove the following result.

Theorem 3.2 *Let $\alpha < \beta$. Consider the perturbed semi-infinite Jacobi matrix J_θ defined in the previous lemma and let I, $\xi(\alpha)$ and $\eta(\beta)$ be defined as in that lemma. If*

$$\frac{2}{N}\sin\left(\pi(1-j/N)\right)\int_\xi^\eta \frac{\mathrm{d}y}{y^2+1} \;>\; \frac{2}{\pi}\left|I\cap\Lambda_M\right|\int_0^{\frac{1}{2M}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)} \frac{\mathrm{d}y}{y^2+1}\,, \tag{3.3}$$

then

$$\int_\alpha^\beta \rho_\theta\left(I\cap\Lambda_M^C\right)\mathrm{d}\theta \;>\; 0\,.$$

P r o o f. The upper bound given by Lemma 2.2 together with the lower bound of Lemma 3.1 and the hypotheses of the theorem give us

$$\int_\alpha^\beta \rho_\theta(I)\,\mathrm{d}\theta \;>\; \int_\alpha^\beta \rho_\theta(I\cap\Lambda_M)\,\mathrm{d}\theta$$

and since

$$\int_\alpha^\beta \rho_\theta(I)\,\mathrm{d}\theta \;=\; \int_\alpha^\beta \rho_\theta\left(I\cap\Lambda_M^C\right)\mathrm{d}\theta + \int_\alpha^\beta \rho_\theta(I\cap\Lambda_M)\,\mathrm{d}\theta$$

we obtain

$$\int_\alpha^\beta \rho_\theta\left(I\cap\Lambda_M^C\right)\mathrm{d}\theta \;>\; 0$$

and the theorem is proved. □

Remark 3.3 If we consider the case $M = 0$ then Theorem 3.2 provides a condition for the existence of mixed spectrum for J_θ. The set Λ_0 is an essential support of the absolutely continuous part of the measure (see [2]), invariant under rank one perturbation. Thus one has mixed spectrum in I for a positive measure set of the λ's in (α,β) if $\int_\alpha^\beta \rho_\theta\left(I\cap\Lambda_0^C\right)\mathrm{d}\theta > 0$ and $|I\cap\Lambda_0| > 0$, Λ_0 dense in I (cf. Remark 2.8).

In more general situations than the ones considered in the theorem, the result stated in it may not be true. To see this let us recall the following statement which appeared in [6]:

> **Theorem (1.2)** *For any measurable set $B \subset \mathbb{R}$ there exists a family of rank-one perturbations $\{A_\lambda\}_{\lambda\in\mathbb{R}}$ such that A_λ has dense absolutely continuous and dense singular spectrum for almost every $\lambda \in B$ and dense absolutely continuous (but no singular) spectrum for almost every $\lambda \notin B$.*

In this result the A_λ's are self-adjoint operators with simple spectrum and therefore, by a theorem of Stone [15], they are self-adjoint Jacobi operators. If we choose B to be a bounded subset of $\mathbb{R}$, then there will not exist an $\ell > 0$ such that any interval (α,β) with at least length ℓ will contain a set of coupling constants of positive measure which correspond to mixed spectra. This illustrates the case opposite to the following examples, where we use Theorem 3.2.

Examples 3.4 (a) In (3.3) take $j = 2$, $N = 4$ and $M = 0$. In this case the condition

$$\int_\alpha^\beta \frac{\mathrm{d}y}{y^2+1} \;>\; 2\,|I\cap\Lambda_0|$$

where $I = \left(-\sqrt{2},\sqrt{2}\right)$, implies singular spectrum in I for a set of coupling constants of positive measure which lies in the interval (α,β).

(b) Let us take a set $B \subset [-2,2] = I$ with $|B| = 1$ such that B and B^C are essentially dense in I, that is, for every subinterval $J \subset I$ we have $|B\cap J| > 0$ and the same for B^C. Now consider the measure $\mathrm{d}\rho(x) = \chi_B(x)\,\mathrm{d}x$ where χ denotes the characteristic function of B ($\chi_B(x) = 1$ if $x \in B$ and zero otherwise).

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Once we have the measure ρ, we can construct a Jacobi Matrix J_ρ such that ρ is the spectral function of J_ρ. In fact if we orthonormalize the monomials $1, \lambda, \lambda^2, \lambda^3, \ldots$ with respect to the scalar product

$$\langle f, g\rangle_\rho \;=\; \int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}\rho(x)$$

to obtain the orthonormal family of polynomials $1 \equiv P_0(\lambda),\ P_1(\lambda),\ \ldots,\ P_n(\lambda),\ \ldots$, where P_n is of degree n, then the coefficients a_k and b_k of J_ρ can be written as

$$\begin{aligned} a_k &= \langle xP_k(x), P_k(x)\rangle_\rho\,,\\ b_k &= \langle xP_k(x), P_{k+1}(x)\rangle_\rho \end{aligned}$$

(see [3]).

From [5, Example 1] we know that $J_\theta := J_\rho + \theta\langle\delta_0,\cdot\rangle$ has mixed spectrum for every $\theta \neq 0$ in every subinterval of I.

If we add a finite rank perturbation R such that the Jacobi Matrix

$$J_R \;=\; J_\rho + R$$

satisfies $a_i = 0$ and $b_i = 1$ for $i = 1, 2, \ldots, N-2$, then the absolutely continuous part of J will be unitarily equivalent to the absolutely continuous part of J_ρ by Kato–Rosenblum theorem see [10, Chapter 10, Theorem 4.3].

Therefore a support for the a.c. spectrum of J_R^θ is B, for every $\theta \in \mathbb{R}$, and the restriction of the spectral measure of J_R^θ is purely singular

$$J_R^\theta \;=\; J_R + \theta\langle\delta_0,\cdot\rangle\delta_0\,.$$

Since $\left|B^C\right| > 0$, it follows from (2.1) the existence of a set C of positive Lebesgue measure such that if $\theta \in C$, then J_N^θ has singular spectrum somewhere in $\mathbb{R}$ (see [5, Cororally 2.7] for example).

Using Theorem 3.2 above, much more can be said about the set C and the location of singular spectrum. If an interval I of the type considered in the theorem is fixed, then we obtain an estimate on the length of (α,β) which implies $|(\alpha,\beta)\cap C| > 0$. This estimate depends on the size of B. Moreover for $\theta \in (\alpha,\beta)\cap C$ the operator J_R^θ will have singular spectrum in I.

Remarks 3.5 1) Theorem 3.2 allows us to conclude existence of singular spectrum in some intervals I. However it does not tell us that this kind of spectra is everywhere in I. To give conditions which guarantee this is an open question as far as we know.

2) The examples given above use only a particular case of Theorem 3.2, namely when $M = 0$.

4 Singular spectrum in the general case

Now we turn to the more general case of an abstract self-adjoint operator A and its family of rank one perturbations

$$A_\theta \;=\; A + \theta\langle\varphi,\cdot\rangle\varphi\,,\quad \theta\in\mathbb{R}\,,\quad \|\varphi\| \;<\; \infty\,,$$

where φ is a cyclic vector for A.

Using just Lemma 2.1 it is possible to give a simple criterion to have singular spectrum somewhere in $\mathbb{R}$.

From Lemma 2.1 we have, if $0 < \alpha < \beta$,

$$\begin{aligned} \int_\alpha^\beta \rho_\theta(\mathbb{R})\,\mathrm{d}\theta \;&=\; \frac{1}{\pi}\int_{\mathbb{R}} \arg(1+\beta F(E+i0)) - \arg(1+\alpha F(E+i0))\,\mathrm{d}E\\ &=\; |A_{\alpha\beta}| + \frac{1}{\pi}\int_\Lambda f_{\alpha\beta}(E)\,\mathrm{d}E\,, \end{aligned} \tag{4.1}$$

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where

$$A_{\alpha\beta} := \left\{E : -\frac{1}{\alpha} < F(E+i0) < -\frac{1}{\beta}\right\},$$
$$\Lambda := \{E : \operatorname{Im} F(E+i0) > 0\},$$
$$f_{\alpha\beta}(E) = \arg(1+\beta F(E+i0)) - \arg(1+\alpha F(E+i0)),$$

and as before

$$F(z) = \left\langle \varphi, (A-z)^{-1}\varphi \right\rangle .$$

Observe that Λ supports the a.c. part of A_θ for every θ and $A_{\alpha\beta}$ supports the singular part of A_θ for a.e. $\theta \in (\alpha, \beta)$. See [13] or [4] for example.

Since $\rho_\theta(\Delta) = \langle E_\theta(\Delta)\varphi, \varphi\rangle$ we know that $\rho_\theta(\mathbb{R}) = \|\varphi\|^2$ for all $\theta \in \mathbb{R}$ and from (4.1) we obtain

$$|A_{\alpha\beta}| = (\beta-\alpha)\,\|\varphi\|^2 - \frac{1}{\pi}\int_\Lambda f_{\alpha\beta}(E)\,\mathrm{d}E \geq (\beta-\alpha)\,\|\varphi\|^2 - |\Lambda| . \tag{4.2}$$

The next result then follows

Theorem 4.1 *If*

$$\beta - \alpha > \frac{|\Lambda|}{\|\varphi\|^2}$$

then the family of rank one perturbations

$$A_\theta = A + \theta\langle\varphi, \cdot\rangle\varphi$$

has some singular spectrum for $\theta \in B$, where $|B| > 0$ and $B \subset (\alpha, \beta)$.

Proof. The condition $\beta - \alpha > \frac{|\Lambda|}{\|\varphi\|}^2$ implies $|A_{\alpha\beta}| > 0$ by (4.2). Since $\int_\alpha^\beta \rho_\theta(A_{\alpha\beta})\,\mathrm{d}\theta = |A_{\alpha\beta}|$ and $A_{\alpha\beta}$ supports the singular part, the results follows. $\square$

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References

[1] N. I. Akhiezer, The Classical Moment Problem and Some Related Questions in Analysis (Oliver and Boyd Ltd., Edinburgh – London, 1965) (translation).

[2] N. Aronszajn, On a Problem of Weyl in the theory of singular Sturm–Liouville equations, Amer. J. Math. **79**, 597–610 (1957).

[3] Ju. M. Berezanskiĭ, Expansions in Eigenfunctions of Selfadjoint Operators, Translations of Mathematical Monographs Vol. 17 (Amer. Math. Soc., Providence, RI, 1968).

[4] R. del Rio, B. Simon, and G. Stolz, Stability of spectral types for Sturm–Liouville operators, Math. Res. Lett. **1**, 437–450 (1994).

[5] R. del Rio, S. Fuentes, and A. Poltoratski, Families of spectral measures with mixed types, Oper. Theory Adv. Appl. **132**, 131–140 (2002).

[6] R. del Rio, S. Fuentes, and A. Poltoratski, Coexistence of spectra in rank-one perturbation problems, Bol. Soc. Mat. Mexicana (3) **8**, 49–61 (2002).

[7] R. del Rio and O. Tchebotareva, Boundary conditions of Sturm–Liouville operators with mixed spectra, J. Math. Anal. Appl. **288**, 518–529 (2003).

[8] G. Freud, Orthogonal Polynomials (Pergamon Press Ltd., Oxford, 1971).

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[9] D. J. Gilbert and D. B. Pearson, On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators, J. Math. Anal. Appl. **128**, No. 1, 30–56 (1987).
[10] T. Kato, Perturbation Theory for Linear Operators, Grundlehren der Mathematischen Wissenchaften Bd. 132 (Springer, New York – Berlin, 1976).
[11] S. Kotani, Lyapunov exponents and spectra for one-dimensional random Schrödinger operators, Contemp. Math. **50**, 277–286 (1986).
[12] B. Simon, Bounded eigenfunctions and absolutely continuous spectra for one-dimensional Schrödinger operators, Proc. Amer. Math. Soc. **124**, No. 11, 3361–3369 (1996).
[13] B. Simon, Spectral analysis of rank one perturbations and applications, CRM Proc. Lecture Notes **8**, 109–149 (1995).
[14] B. Simon, The classical moment problem as a self-adjoint finite difference operator, Adv. Math. **137**, 82–203 (1998).
[15] M. H. Stone, Linear Transformations in Hilbert Space and their Applications to Analysis, American Mathematical Society Colloquium Publications Vol. 15 (Amer. Math. Soc., Providence, RI, 1932).

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